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Analyticity Properties and a Convergent Expansion for the Inverse Correlation Length of the High-Temperature d-Dimensional Ising Model

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We show that the inverse correlation length $m(\beta)$ (= mass of the fundamental particle of the associated lattice quantum field theory) of the spin–spin correlation function $\langle s_x s_y \rangle$, $x, y \in Z^d$, of the d-dimensional Ising model admits the representation

$$m(\beta) = -\ln \beta + r(\beta)$$

for small inverse temperatures $\beta > 0$. $r(\beta)$ is a d-dependent function, analytic at $\beta = 0$. c_n, the nth $\beta = 0$ Taylor series coefficient of $r(\beta)$ can be computed explicitly from the Z^d limit of a finite number of finite lattice Λ spin–spin correlation functions $\langle s_0 s_x \rangle_\Lambda$ for a finite number of $x = (x_1, x_2, \ldots, x_d)$, $|x| = \sum_{i=1}^d |x_i| < R(n)$, where $R(n)$ increases with n. Furthermore, there exists a $\beta' > 0$, such that for each $\beta \in (0, \beta') m(\beta)$ is analytic. Similar results are also obtained for the dispersion curve $\omega(\mathbf{p})$, $\omega(\mathbf{p}) \geqq \omega(\mathbf{0}) = m$, $\mathbf{p} \in (-\pi, \pi]^{d-1}$, of the fundamental particle of the associated lattice quantum field theory.

KEY WORDS: Ising model; correlation length; correlation function; expansion for correlation length; analyticity of correlation length; high-temperature Ising model.

1. INTRODUCTION

In this paper we give analyticity properties and a convergent expansion for the inverse correlation length $m(\beta)$ of the classical spin ± 1 nearest-neighbor ferromagnetic Ising model on a d-dimensional Z^d lattice in the region of small $|\beta|$, β the inverse temperature. Equivalently $m(\beta)$ is the

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mass of the fundamental particle of the associated Ising lattice quantum field theory.[2] For $\beta > 0$, $m(\beta)$ is defined by

$$m(\beta) = \lim_{x_1 \to \infty} \frac{-1}{x_1} \ln\langle s_0 s_{x=(x_1,\mathbf{0})}\rangle$$

$\langle s_x s_y\rangle \equiv G(x; y, \beta) = \lim_{\Lambda \uparrow Z^d}(G_\Lambda(x; y, \beta) \equiv \langle s_x s_y\rangle_\Lambda)$, $x, y \in Z^d$ where $G_\Lambda(x; y, \beta)$ is the spin–spin correlation function (cf) in the Gibbs ensemble with Boltzmann factor $\exp[(\beta/2)\sum_{x,y \in \Lambda} s_x s_y]$ for the finite lattice $\Lambda \subset Z^d$ at inverse temperature $\beta > 0$ and we denote points $x \in \Lambda$ by $x = (x_1, x_2, \ldots, x_d) = (x_1, \mathbf{x})$ and $|x| = \sum_{i=1}^d |x_i|$, $|\mathbf{x}| = \sum_{i=2}^d |x_i|$.

It is a consequence of high-temperature statistical mechanics expansions, say, the polymer expansion in Ref. 3, that the n-pt. cf are analytic for small $|\beta|$ and translation invariant so that $G(x; y, \beta) = G(x-y, \beta)$. By Griffiths inequalities and iteration methods[4] one can obtain upper and lower bounds on $\langle s_0 s_x\rangle$ which imply the asymptotic result $\lim_{\beta \downarrow 0}[m(\beta)/-\ln\beta] = 1$.

However, the analyticity properties of $m(\beta)$ are not clear. Spectral analysis methods of quantum field theory have been used in Ref. 1 to obtain an asymptotic expansion of $m(\beta)$ to order β and to show that there is an upper mass gap and isolated dispersion curve

$$\omega(\mathbf{p}) \equiv \lim_{x_1 \to \infty} \frac{-1}{x_1} \ln\Big(\sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\langle s_0 s_{(x_1,\mathbf{x})}\rangle\Big) \geqq \omega(\mathbf{0}) = m$$

where $\mathbf{p} \in (-\pi, \pi)^{d-1}$ is the momentum. Also, graphical methods were employed by Ref. 5 for real β to obtain the asymptotic formula of Ref. 1. Extrapolations in β of similar expansions have been used in lattice gauge theory in Ref. 6 as an alternative to Monte Carlo methods to detect and locate critical points, for example, where $m(\beta) = 0$.

From an inspection of explicit formulas for $m(\beta)$ for $d = 1$ and 2 (in $d = 1$ $\langle s_0 s_{x_1}\rangle = (\tanh\beta)^{x_1}$, $m(\beta) = -\ln\tanh\beta$) we see that $m(\beta) = -\ln\beta + r(\beta)$ where $r(\beta)$ is a dimension-dependent function analytic at $\beta = 0$. We note that $\langle s_0 s_x\rangle$ is an entire function of β but the radius of convergence of the $\beta = 0$ Taylor series of $r(\beta)$ is $|\beta| = \pi/2$ in $d = 1$ $[r(\beta) = -\ln(\mathrm{th}\,\beta/\beta)]$; however, for each $\beta \in (0, \infty)$ $m(\beta)$ is analytic. In this paper we show similar results in d dimensions, i.e.,

(1) $m(\beta) = -\ln\beta + r(\beta)$, $\beta > 0$ small, $r(\beta)$ analytic at $\beta = 0$,

(2) there exists a $\beta' > 0$ such that for each $\beta \in (0, \beta')$ $m(\beta)$ is analytic.

Furthermore we show that

(3) $c_n = (1/n!)(d^n r/d\beta^n)(\beta = 0)$, the nth $\beta = 0$ Taylor series coefficient of $r(\beta)$, can be computed explicitly from the Z^d limit of a finite number of $\beta = 0$ Taylor series coefficients of the finite lattice cf $\langle s_0 s_x\rangle_\Lambda$ for a finite number of x, $|x| < R(n)$ where $R(n)$ increases with n.

The second result is an immediate consequence of results of Ref. 1 and the analytic implicit function theorem[7,8]; the first result also depends on many results of Ref 1. The method used here is that of quantum field theory; we obtain an implicit equation for $m(\beta)$ as the zero $p_1 = im(\beta)$, $\mathbf{p} = 0$ $[m(\beta) > 0]$ of the Fourier transform $\tilde{\Gamma}(p_1, \mathbf{p})$ of the convolution inverse $\Gamma(x, y)$ in $l_2(Z^d)$ of $G(x, y)$ where $G(x, y)$ is interpreted as a matrix operator in $l_2(Z^d)$. For the relation of this zero to the previous definition and to the spectrum of the associated lattice quantum field theory see Refs. 1 and 2.

The organization of this paper is as follows: In Section 2 we collect some results from Ref. 1 needed to prove (1), give some extensions and prove (2). In Section 3 we prove (1); in Section 4 we prove (3). Section 5 is devoted to discussion and some open questions. In an appendix we give some estimates used in the proof of (1).

2. β ANALYTICITY OF $m(\beta)$ IN $(0, \beta')$

We state some results from Ref. 1 in the form needed here. $\tilde{G}(p)$ or $\tilde{G}(p, \beta) = [1/(2\pi)^{d/2}]\sum_{x \in Z^d} e^{ipx} G(x, \beta)$, $p \in R^d$, $px = \sum_{i=1}^d p_i x_i$, denotes the Fourier transform. Similarly for $\tilde{\Gamma}(p)$, the Fourier transform of the convolution inverse $\Gamma(x; \beta) \equiv \Gamma(y; z, \beta)$, $y - z = x$, of G in $l_2(Z^d)$. We denote by $\|G\|$ $(\|\Gamma\|)$ the $l_2(Z^d)$ operator norm of G (Γ) considered as matrix operators.

Lemma 2.1.[1] (a) There exist $c, c_1, c_2 > 0$ and $\beta_0 > 0$ such that for all $\beta \in C$, $|\beta| < \beta_0$ $G(x)$ is analytic in β, $|G(x)| \leqq c_1 |c\beta|^{|x_1| + |\mathbf{x}|}$, $\|G\| < c_2$, $G(x_1, \mathbf{x}) = G(-x_1, \mathbf{x})$.

(b) There exist β_1 such that for all $\beta \in C$, $|\beta| < \beta_1$ $\tilde{G}(p_1, \mathbf{p})$ is analytic in β and p in $|\mathrm{Im}\, p_i| < -\ln|\beta/\beta_1|$, $i = 1, 2, \ldots, d$.

(c) For p real, $\beta \in C$, $|\beta|$ small $\tilde{G}$ has the $\beta = 0$ Taylor expansion

$$\tilde{G}(p, \beta) = 1 + 2\beta \sum_{i=1}^{d} \cos p_i + \beta^2 \int_0^1 (1 - t) \sum_x e^{ipx} \frac{\partial^2 G}{\partial \xi^2}(x, \xi = \beta t)\, dt$$

(d) For β real, $\lim_{\beta \downarrow 0} [m(\beta) / -\ln \beta] = 1$.

Remark. See the Appendix for the proof of (a).

Lemma 2.2.[1] (a) There exist c, c_1, $\beta_5 > 0$ such that for all $\beta \in C$, $|\beta| < \beta_5$ $\Gamma(x)$ is analytic in β and $|\Gamma(x)| \leqq c|\beta/\beta_5|^{2|x_1| + |\mathbf{x}|}$, $x \neq (\pm 1, \mathbf{0})$; for $x = (\pm 1, \mathbf{0})$ replace 2 by 1. $|\Gamma| < c_1$, $\Gamma = (1 + Q)^{-1} = \sum_{n=0}^{\infty} (-1)^n Q^n$, $Q(x, y) = G(x; y) - \delta_{xy}$, is convergent in operator norm. Also $\Gamma(x_1, \mathbf{x}) = \Gamma(-x_1, \mathbf{x})$.

(b) There exist $\beta_6 > 0$ such that for all $\beta \in C$, $|\beta| < \beta_6$, $\tilde{\Gamma}(p)$ is analytic in $|\mathrm{Im}\, p_1| < -2\ln|\beta/\beta_6|$, $|\mathrm{Im}\, p_i| < -\ln|\beta/\beta_6|$, $i = 2, \ldots, d$.

(c) $\tilde{G}(p)\,\tilde{\Gamma}(p) = 1$ holds in the region of analyticity of $\tilde{G}(p)$.

(d) For p real, $\beta \in C$, $|\beta|$ small, $\tilde{\Gamma}(p)$ has the $\beta = 0$ Taylor expansion

$$\tilde{\Gamma}(p) = 1 - 2\beta \sum_{i=1}^{d} \cos p_i + \beta^2 \int_0^1 (1-t)\, \frac{\partial^2 \tilde{\Gamma}}{\partial \xi^2}(p, \xi = \beta t)\, dt$$

Remarks. (1) Note the faster falloff of $\Gamma(x)$ in $|x_1|$ and consequent larger p_1 analyticity region of $\tilde{\Gamma}(p)$ as compared to that of $G(x)$ and $\tilde{G}(p)$. For $x \neq (\pm 1, \mathbf{0})$ the 2 in (a) can be replaced by 3 as shown in the Appendix. In (b) $-2\ln$ can be replaced by $-3\ln$.

(2) The constant and linear term of the Taylor expansion for $\tilde{\Gamma}(p)$ are determined from Lemmas 2.1c and 2.2c.

Lemma 2.3.[1,2] For $\beta > 0$ and small and for each $\mathbf{p} \in (-\pi, \pi]^{d-1}$, (a) there exists a dispersion curve $\omega(\mathbf{p})$, real analytic in $\mathbf{p}$, defined by $\tilde{\Gamma}(p_1 = i\omega(\mathbf{p}), \mathbf{p}) = 0$, $\omega(\mathbf{p}) > 0$. $p_1 = i\omega(\mathbf{p})$ is the only zero of $\tilde{\Gamma}(p_1, \mathbf{p})$ in $0 < \mathrm{Im}\, p_1 - 2\ln|\beta/\beta_6|$, $|p_i| < \pi$, $i = 2, 3, \ldots, d$, and is simple. Furthermore $\omega(\mathbf{p}) \geqq \omega(\mathbf{0}) = m$ and $\lim_{\beta \downarrow 0}(\omega(\mathbf{p})/m) = 1$ uniformly in $\mathbf{p} \in (-\pi, \pi]^{d-1}$; (b) there exists a $Z'(\mathbf{p}) > 0$, real analytic in $\mathbf{p}$, such that $(\partial\tilde{\Gamma}/\partial p_1)(p_1 = i\omega(\mathbf{p}), \mathbf{p}) = Z'(\mathbf{p}) > 0$.

We have the following:

Theorem 2. There exists a $\beta' > 0$ such that for each $\beta \in (0, \beta')$ $m(\beta)$ is analytic.

Proof. With $\mathbf{p}$ fixed $\tilde{\Gamma}(p_1, \mathbf{p}, \beta)$ is jointly analytic in p_1 and β for $\beta \in (0, \beta')$, $p_1 = i\omega(\mathbf{p})$ by Lemma 2.2b. From Lemma 2.3b $(\partial\tilde{\Gamma}/\partial p')(p_1 = i\omega(\mathbf{p}), \mathbf{p}, \beta) = Z'(\mathbf{p}) > 0$, so by the analytic implicit function theorem[7,8] $\omega(\mathbf{p})$ is analytic in β. ■

In the next section we will need to control the remainder term of $\tilde{\Gamma}(p)$ in Lemma 2.2d. Define

$$\Gamma_s(x, \beta) = \Gamma(x, \beta) - \sum_{m=0}^{1} \frac{\beta^m}{m!} \frac{\partial^m \Gamma}{\partial \beta^m}(x, \beta = 0)$$

$$= \beta^2 \int_0^1 (1-t)\, \frac{\partial^2 \Gamma}{\partial \xi^2}(x, \xi = \beta t)\, dt$$

and for $n = 0, 1, \ldots$ $\Gamma_s(n, \beta) \equiv \sum_{\mathbf{x}} \Gamma_s(x_1 = n, \mathbf{x}, \beta)$. We have the following:

Lemma 2.4. There exist c, c_3, c_4 and $\beta_7 > 0$ such that for all $|\beta| < \beta_7$, (a) $|\Gamma_s(x, \beta)| \leqq c_3|\beta|^2$, (b) $\Gamma_s(n, \beta)/\beta^n$ is analytic, $|\Gamma_s(n, \beta)| \leqq c_4|c\beta|^{2n}$ for $n \neq 0$, $|\Gamma_s(0, \beta)| \leqq c_4|\beta|^2$.

Remark. For $x = (\pm 1, \mathbf{0})$, $|\Gamma_s(x, \beta)| \leqq c|\beta|^3$ holds. For $n \neq 0$, 2 can be replaced by 3 in (b).

Proof. (a) Follows from Lemma 2.2a and a Cauchy estimate on $(\partial^2\Gamma/\partial\xi^2)(x, \xi = \beta t)$. (b) From Lemma 2.2a for x such that $2|x_1| + |\mathbf{x}| \geqq 2$, $x \neq (\pm 1, \mathbf{0})$, $\Gamma_s(x, \beta) = \Gamma(x, \beta)$, $|\Gamma_s(x, \beta)| \leqq c|\beta/\beta_5|^{2|x_1| + |\mathbf{x}|}$ and using (a) for the other x the result follows on summing these bounds over $\mathbf{x}$ with $x_1 = n$. ■

3. ANALYTICITY PROPERTIES OF $m(\beta)$ AT $\beta = 0$

We assume $\mathbf{p} = 0$ throughout and write $\tilde{\Gamma}(p_1, \beta)$ for $\tilde{\Gamma}(p_1, \mathbf{p} = 0, \beta)$. From Lemma 2.2d, p_1 real, $|\beta|$ small,

$$\tilde{\Gamma}(p_1, \beta) = 1 - 2\beta \cos p_1 - 2\beta(d-1) + \sum_{x_1} e^{ip_1x_1}\left[\sum_{\mathbf{x}} \Gamma_s(x, \beta)\right]$$

and by the estimates of Lemma 2.4 the right side provides an analytic extension in p_1 to $|\operatorname{Im} p_1| < -2\ln|c\beta|$, the $x_1, \mathbf{x}$ series converging absolutely. As $\Gamma(x_1, \mathbf{x}) = \Gamma(-x_1, \mathbf{x})$ we can write

$$\begin{aligned}\tilde{\Gamma}(p_1, \beta) = {} & 1 - \beta(e^{-ip_1} + e^{ip_1}) - 2\beta(d-1) + \Gamma_s(0, \beta) \\ & + \sum_{n=1}^{\infty} \Gamma_s(n, \beta)(e^{ip_1n} + e^{-ip_1n})\end{aligned}$$

Considering the asymptotic form of m, i.e., $m(\beta) \simeq -\ln \beta$, we see that e^{-ip_1} is not well behaved at $p_1 = im(\beta)$ for small β which motivates the introduction of the auxiliary complex variable v and function $H(v, \beta)$ such that $H(v = \beta e^{-ip_1} - 1, \beta) = -\tilde{\Gamma}(p_1, \beta)$ where

$$\begin{aligned}H(v, \beta) = {} & v + \frac{\beta^2}{1+v} + 2\beta(d-1) - \Gamma_s(0, \beta) \\ & - \sum_{n=1}^{\infty} \Gamma_s(n, \beta)\left[\frac{(1+v)^n}{\beta^n} + \frac{\beta^n}{(1+v)^n}\right]\end{aligned}$$

The domain of H will be specified below and we will show that indeed H is a nice function to which the analytic implicit function theorem can be applied.

Lemma 3.1. For all $\beta \in C$, $|\beta|$ sufficiently small, (a) $H(v, \beta)$ extends to a function jointly analytic in a region R which contains a product of open disks D_v, D_β centered at $v = 0$, $\beta = 0$, respectively. The infinite series converge uniformly and absolutely in R.

(b) $H(v = 0, \beta = 0) = 0$ and $(\partial H/\partial v)(v, \beta) \neq 0$ in R.

(c) There exist complex neighborhoods N_β of $\beta = 0$, N_v of $v = 0$ and a unique analytic function $v(\beta)$, $v(0) = 0$, such that $H(v(\beta), \beta) = 0$ in N_β.

Furthermore $v(\beta)$ admits the explicit integral representation

$$v(\beta)=\frac{1}{2\pi i}\int_C \frac{w(\partial H/\partial w)(w,\beta)}{H(w,\beta)}\,dw$$

where C is the positively oriented contour $|w|=\rho$, ρ sufficiently small.

We defer the proof of Lemma 3.1 to the end of this section and now give our main result concerning $m(\beta)$.

Theorem 3. For all β real, $\beta\in(0,\beta'')$, β'' sufficiently small $m(\beta) = -\ln\beta + r(\beta)$, $r(\beta)=\ln(1+v(\beta))$. Let N_0 be a sufficiently small complex neighborhood of $\beta=0$. Then $r(\beta)$ provides an analytic continuation of $m(\beta)-(-\ln\beta)$ to N_0; of $m(\beta)$ to N_0 with the negative real β axis deleted.

Remark. Similar results hold for the dispersion curve $\omega(\mathbf{p})$ by not setting $\mathbf{p}=0$ in $\tilde{\Gamma}(p_1,\mathbf{p},\beta)$.

Proof. By Lemma 2.3a for $\beta>0$ and sufficiently small there is only one zero of $\tilde{\Gamma}(p_1,\beta)$ in $0<\operatorname{Im}p_1<-2\ln|c\beta|$ and it is at $p_1=im(\beta)$, $0<m(\beta)<-(1+\epsilon)\ln|\beta c|$, $\epsilon>0$. For β sufficiently small the zero of $H(v(\beta),\beta)$ is in the analyticity region of $\tilde{\Gamma}$ as $v(\beta)=O(\beta)$ so the zero of H is $p_1=im(\beta)$. Thus for β real, $\beta\in(0,\beta'')$, β'' sufficiently small,

$$e^{m(\beta)}=(1/\beta)\left[1+v(\beta)\right]$$

or $m(\beta)=-\ln\beta+\ln[1+v(\beta)]$. ∎

Proof of Lemma 3.1. (a) For $|\beta|$ small and $|v|<1$ each term of $H(v,\beta)$ is analytic using Lemma 2.4b. Again, using Lemma 2.4b the infinite series in $H(v,\beta)$ converge uniformly in β, v, $|\beta|,|v|$ small by the ratio test.

(b) By Lemma 2.4b and (a) $H(0,0)=0$. We now examine

$$\frac{\partial H}{\partial v}(v,\beta)=1-\frac{\beta^2}{(1+v)^2}-\sum_{n=1}^{\infty}\Gamma_s(n,\beta)\left[\frac{-n\beta^n}{(1+v)^{n+1}}+\frac{n(1+v)^{n-1}}{\beta^n}\right]$$

By Lemma 2.4b and using the ratio test $\lim_{\beta\to 0}(\partial H/\partial v)(v,\beta)=1$ uniformly for $|v|$ small.

(c) follows from the analytic implicit function theorem. As H is analytic we are in the pleasant situation where the implicit function becomes explicit.[8] ∎

4. A CONVERGENT EXPANSION FOR $m(\beta)$

From Theorem 3 in Section 3 we see that c_m, the mth $\beta=0$ Taylor series coefficient of $r(\beta)$ is determined from the $\beta=0$ Taylor series coefficients, a_j, $0 \leqq j \leqq m$, of $v(\beta)$. We show below that a_n can be determined explicity from the Z^d limits of a finite number of the computable $\beta = 0$ Taylor series coefficients of $G_\Lambda(0; x, \beta) \equiv \langle s_0 s_x \rangle_\Lambda$, $0, x \in \Lambda \subset Z^d, \Lambda$ finite, for all x, $|x| < R(n)$, where $R(n)$ is an increasing function of n.

The $a_n = (1/n!)(d^n v/d\beta^n)(0)$ are given explicitly in terms of the β and v partial derivatives of $H(v, \beta)$ at $(v, \beta) = (0, 0)$ (see Ref. 7). For example, letting $D_\beta^n \equiv \partial^n/\partial\beta^n$, $D_v^m \equiv \partial^m/\partial v^m$, we have

$$\frac{dv}{d\beta} = -D_\beta H(D_v H)^{-1},$$

$$\frac{d^2v}{d\beta^2} = -\left[D_\beta D_v H D_\beta v + D_\beta^2 H\right](D_v H)^{-1}$$

$$+ D_\beta H(D_v H)^{-1}\left[D_v^2 H D_\beta v + D_v D_\beta H\right]$$

The general expression for $d^s v(\beta)/d\beta^s$ involves a finite number of β and w derivatives of $F(v, \beta)$ which in turn depends on $\Gamma_s(n, \beta)$ and their β derivatives up to a finite order. Using the bounds on $\Gamma_s(n, \beta)$ from Lemma 2.4 it is seen that $d^s v/d\beta^s(\beta = 0)$ depends on only $\{\Gamma_s(n, \beta)\}_{n=0}^m$ at $\beta = 0$, m finite, and their $\beta = 0$ derivatives up to a finite order. We now show that to find $(d^k\Gamma_s/d\beta^k)(n, \beta = 0)$ it suffices to know a finite number of the $\beta = 0$ Taylor series coefficients of $G(x)$ for a finite number of x, $|x| < R$ depending on k and n, which can be computed explicitly. Recall $\Gamma_s(n, \beta) = \sum_{\mathbf{x}} \Gamma_s(x_1 = n, \mathbf{x}, \beta)$,

$$\Gamma_s(n, \mathbf{x}, \beta) \equiv \Gamma(x_1 = n, \mathbf{x}, \beta) - \Gamma(x_1 = n, \mathbf{x}, \beta = 0)$$

$$- \beta \frac{\partial}{\partial\beta} \Gamma(x_1 = n, \mathbf{x}, \beta = 0)$$

and the Neumann series for $\Gamma(x; y) \equiv \Gamma(x - y)$, i.e.,

$$\Gamma(x; y) = \delta_{xy} - Q(x; y) + \sum_z Q(x; z)Q(z; y) + \cdots$$

where $Q(x; y) = G(x; y)(1 - \delta_{xy})$, $G(x; y) \equiv G(x - y)$. Using the falloff of $G(x)$ given by Lemma 2.1 we see that only a finite number of terms contribute to $(d^k\Gamma_s/d\beta^k)(n, \mathbf{x}, \beta = 0)$ and only a finite number of $\mathbf{x}$ contribute to $(d^k\Gamma_s/d\beta^k)(n, \mathbf{x}, \beta = 0)$.

Using the analyticity in β and the uniform convergence in β of $\lim_{\Lambda \uparrow Z^d} G_\Lambda(0; y, \beta) = G(0; y, \beta)$ and its β derivatives the $\beta = 0$ derivatives of $G(y, \beta)$ are given by the Z^d limits of $\beta = 0$ derivatives of $G_\Lambda(0; y, \beta)$, $0, y \in \Lambda \subset Z^d$.Finally, if so desired, the $\beta = 0$ derivatives of $G_\Lambda(0; y, \beta)$ can be obtained from the quotient $N_\Lambda(\beta)/D_\Lambda(\beta) \equiv G_\Lambda(0; y, \beta)$ [$D_\Lambda(\beta)$ is the partition function], separately expanding $N_\Lambda(\beta)$ and $D_\Lambda(\beta)$ [$D_\Lambda(\beta)$ is analytic at $\beta = 0$ for finite Λ] as can be done, for example, in establishing the β expansion of $\tilde{G}(p, \beta)$ of Lemma 2.1c.

5. CONCLUDING DISCUSSION

Similar methods apply to other lattice models, such as the Ising with a magnetic field, the $P(\phi)$ models,[9] with or without a magnetic field, pure gauge and gauge–Higgs models. In the pure gauge model the order of the expansion of $\tilde{\Gamma}$ is different.

It would be interesting to develop a more direct method for obtaining the coefficients of the expansion of $r(\beta)$. For example, Cauchy majorants[10] may be appropriate. The possible existence of other expansions which are simpler and more direct should also be investigated.

We raise the question of low temperature analyticity. The "right" variable for analyticity should be an activity, like $z = e^{-\beta}$ rather than $y = 1/\beta$ which gives an essential singularity at $\beta = \infty$. This also suggests that an activity-type variable may be appropriate for analyticity or Borel summability in other problems such as the double well,[11] large distance interaction between atoms and expansions about mean field theory.[12,13]

A convergent perturbation theory for the mass of the time-continuum *infinite* space lattice Hamiltonian version of lattice spin, gauge, and gauge-matter lattice models has yet to be developed.

APPENDIX

Here we obtain some bounds on G and Γ in Lemmas 2.1 and 2.2 extending these of Ref. 1. Many of the proofs are patterned after Ref. 14. It is convenient to introduce the complex coupling parameters $\{w_p\}, z$ as follows: w_p replaces β for all bonds $B_p^{\parallel}$, the bonds parallel to the 1-direction between the hyperplanes $k_1 = p$ and $k_1 = p + 1$; z replaces β for all bonds $B^{\perp}$, the bonds perpendicular to the 1-direction. We write the complexified expectation in the finite volume Λ, for a local function F, as

$$\langle F \rangle_\Lambda = Z_\Lambda^{-1} 2^{-|\Lambda|} \sum_{\{s\}} F(s) \exp\left\{ \sum_{p=-n}^{n-1} w_p \sum_{x,y \in B_p^{\parallel}} s_x s_y + z \sum_{x,y \in B^{\perp}} s_x s_y \right\}$$

where Λ extends from $-n$ to n in the 1-direction. By the polymer expansion of Ref. 3 $\langle F \rangle_\Lambda$ is analytic for $\{|w_p|\}, |z|$ small, uniform in Λ. The bounds we will obtain are uniform in Λ and carry over to the $\Lambda \to Z^d$ limit. We let $G_\Lambda(i, j, \{w_p\}, z)$ and $\Gamma_\Lambda(i, j, \{w_p\}, z)$ denote the 2-pt. function and its convolution inverse for volume Λ where $G_\Lambda, \Gamma_\Lambda$ are also interpreted as convolution operators in $l_2(\Lambda)$. Let e_1 denote the unit vector in the positive 1-direction.

Lemma A1. For $\{|w_p|\}, |z|$ small,

(a) $$G_\Lambda(i, j, \{w_p\}, z)|_{w_p=0} = 0, \qquad i_1 \leqq p < j_1$$

(b) $$\frac{\partial^m G}{\partial z^m} \Lambda(i, j, \{w_p\}, z)|_{z=0} = 0, \qquad 0 \leqq m < |\mathbf{j} - \mathbf{i}|$$

Corollary A1. For $i_1 \leqq p < j_1$,

$$G_\Lambda(i, j, \{w_p\}, z) = \prod_{p=i_1}^{j_1-1} w_p z^{|\mathbf{j}-\mathbf{i}|} F_\Lambda(i, j, \{w_p\}, z)$$

F_Λ analytic; for $j_1 = i$, drop the w_p factors and replace F_Λ by K_Λ, K_Λ analytic.

Proof of Lemma A1. (a) The numerator of $G_\Lambda|_{w_p=0}$ factorizes and the sum over $\{s\}$ gives zero.

(b) Expand the exponential of the numerator of G_Λ. Each term must have bounds connecting i and j to give a nonzero contribution. In particular there must be at least $|\mathbf{j} - \mathbf{i}|$ bonds from $B^\perp$ giving a factor of $z^{|\mathbf{j}-\mathbf{i}|}$. ■

Lemma A2. For $\{|w_q|\}, |z|$ small the operator $\Gamma_\Lambda \equiv G_\Lambda^{-1}$ exists, $\|\Gamma_\Lambda\| \leqq 2$ and $\Gamma_\Lambda(i, j, \{w_p\}, z)$ is analytic. Furthermore,

(a) $$\Gamma_\Lambda(i, j, \{w_p\}, z)|_{w_p=0} = 0, \qquad i_1 \leqq p < j_1$$

(b) $$\left.\frac{\partial \Gamma_\Lambda}{\partial w_p}(i, j, \{w_p\}, z)\right|_{w_p=0} = \delta_{i+e_1, j}, \delta_{i_1, p}, \quad i_1 \leqq p < j_1$$

(c) $$\left.\frac{\partial^2 \Gamma_\Lambda}{\partial w_p^2}(i, j, \{i, j, \{w_p\}, z)\right|_{w_p=0} = 0, \qquad i_1 \leqq p < j_1$$

(d) $$\left.\frac{\partial^m \Gamma_\Lambda}{\partial z^m}(i, j, \{w_p\}, z)\right|_{z=0} = 0, \qquad 0 \leqq m < |\mathbf{j} - \mathbf{i}|$$

Corollary A3. For $j \neq i + e_1, j_1 > i_1$

$$\Gamma_\Lambda(i, j, \{w_p\}, z) = \prod_{p=i_1}^{j_1-1} w_p^3 z^{|\mathbf{j}-\mathbf{i}|} H_\Lambda(i, j, \{w_p\}, z)$$

H_Λ analytic; for $j_1 = i_1$ drop the w_p factors and replace H_Λ by L_Λ, L_Λ analytic.

Proof of Lemma A2. Write $G_\Lambda(i, j) = \delta_{ij} + Q_\Lambda(i, j)$, $Q_\Lambda(i, j) = G_\Lambda(i, j)(1 - \delta_{ij})$. By a Cauchy estimate Corollary A1 implies the bound

$$|G_\Lambda| = C^{|j-i|} \prod_{p=i_1}^{j_1-1} |w_p||z|^{|j-i|}$$

and using the bound $\|A\| \leqq (\max_j \sum_k A_{jk}|)^{1/2} (\max_k \sum_j |A_{jk}|)^{1/2}$ for a matrix operator $A : l_2 \to l_2$ it follows that $|Q_\Lambda| < (1/2)$ for $\{|w_p|\}, |z|$ sufficiently small. Thus the Neumann series for Γ_Λ converges.

(a) Lemma A1a implies $G_\Lambda|_{w_p=0}$ reduces the complementary subspaces $l_2(\Lambda_{p\leqq})$ and $l_2(\Lambda_{p>})$ where $\Lambda_{p\leqq}$ $(\Lambda_{p>}) = \{k \in \Lambda \mid k_1 \leqq p\}(\{k \in \Lambda \mid k_1 > p\})$ and the same is true for $\Gamma_\Lambda|_{w_p=0}$.

(b) From $G_\Lambda \Gamma_\Lambda = 1$, $(\partial/\partial w_p)(G_\Lambda \Gamma_\Lambda) = 0$ we find

$$\left.\frac{\partial \Gamma_\Lambda}{\partial w_p}(i, j)\right|_{w_p=0} = -\sum_{k,l} \Gamma_\Lambda(i, k) \frac{\partial G_\Lambda}{\partial w_p}(k, l) \Gamma_\Lambda(l, j)\Big|_{w_p=0}$$

By the reducing subspace property (rsp) of Γ_Λ we can restrict the sum to $k_1 \leqq p < l_1$. A short calculation shows that

$$\left.\frac{\partial G_\Lambda}{\partial w_p}(r, s)\right|_{w_p=0} = \begin{cases} \displaystyle\sum_{m\,:\,m_1=p} G_\Lambda(r, m) G_\Lambda(m + e_1, s)\Big|_{w_p=0}, & r_1 \leqq p < s_1 \\ 0 & \text{otherwise } {}_{w_p=0}, \end{cases}$$

which when substituted in the above gives

$$\begin{aligned} \left.\frac{\partial \Gamma_\Lambda}{\partial w_p}(i, j)\right|_{w_p=0} &= \sum_{\substack{k,l \\ m\,:\,m_1=p}} \Gamma_\Lambda(i, k) G_\Lambda(k, m) G_\Lambda(m + e_1, l) \Gamma_\Lambda(l, j)\Big|_{w_p=0} \\ &= \sum_{m;m_1=p} \delta_{i,m} \delta_{m+e_1,j} = \delta_{i_1,p} \delta_{i+e_1,j} \end{aligned}$$

where we have extended the sum over all k, l by the rsp of Γ_Λ.

(c) Write $\partial^2 \Gamma_\Lambda/\partial wp^2 = I + 2II$, where $I \equiv -\Gamma_\Lambda(\partial^2 G_\Lambda/\partial w_p^2)\Gamma_\Lambda$ and $II \equiv \Gamma_\Lambda(\partial G_\Lambda/\partial w_p)\Gamma_\Lambda(\partial G_\Lambda/\partial w_p)\Gamma_\Lambda$. By expanding the numerator of G_Λ in powers of w_p we find

$$\left.\frac{\partial^2 G_\Lambda}{\partial w_p^2}(r, s)\right|_{w_p=0} = 0, \qquad r_1 \leqq p < s_1$$

which upon substituting in I at $w_p = 0$ gives

$$\left.\left(\Gamma_\Lambda \frac{\partial^2 G_\Lambda}{\partial w_p^2} \Gamma_\Lambda\right)(i, j)\right|_{w_p=0} = \sum_{k,l} \Gamma_\lambda(i, k) \frac{\partial^2 G_\Lambda}{\partial w_p^2}(k, l) \Gamma_\Lambda(l, j)\Big|_{w_p=0}$$

By the rsp of Γ_Λ the sum can be restricted to $k_1 \leqq p$, $l_1 > p$ but then $(\partial^2 G_\Lambda/\partial wp^2)(k,l)|_{w_p=0} = 0$ so the term I gives zero.

Consider the term II at $w_p = 0$. We write

$$II(i,j)|_{w_p=0} = \sum_{k,k',k'',l} \Gamma_\Lambda(i,k)\frac{\partial G_\Lambda}{\partial w_p}(k,k')\Gamma_\Lambda(k',k'')\frac{\partial G_\Lambda}{\partial w_p}(k'',l)\Gamma_\Lambda(l,j)\bigg|_{w_p=0}$$

Since $i_1 \leqq p < j_1$ the sum over k (l) can be restricted to $k_1 \leqq p$ ($l_1 > p$), but then to get a nonzero contribution the sum over k' (k'') can be restricted to $k'_1 > p$ ($k''_1 \leqq p$) by the rsp of $(\partial G_\Lambda/\partial w_p)|_{w_p=0}$ in the proof of (b). Substituting for $(\partial G_\Lambda/\partial w_p)|_{w_p=0}$ and then extending the sums to all k, l by the rsp of Γ_Λ we obtain

$$II(i,j)|_{w_p=0}$$

$$= \sum_{\substack{k':k'_1>p\\ k'':k''_1\leqq p}} \sum_{\substack{m,n:\\ m_1=n_1=p}} \delta_{i,m} G_\Lambda(m+e_1,k')\Gamma_\Lambda(k',k'')G_\Lambda(k'',n)\delta_{n+e_1,j}\bigg|_{w_p=0}$$

By the rsp of $G_\Lambda|_{w_p=0}$ we can extend the sum over all k'' to get

$$II(i,j)|_{w_p=0} = \sum_{k';k'_1>p} \sum_{\substack{m,n:\\ m_1=n_1=p}} \delta_{i,m} G_\Lambda(m+e_1,k')\bigg|_{w_p=0} \delta_{k',n}\delta_{n+e_1,j} = 0$$

since $n_1 = p$, $k'_1 > p$.

(d) Write $l_2(\Lambda) = \bigoplus_{\mathbf{k}} l_2(\Lambda_{\mathbf{k}})$ where $\mathbf{k}$ denotes $\{k \in \Lambda \mid k = (0,\mathbf{k})\}$ and $\Lambda_{\mathbf{k}} = \{k \in \Lambda \mid k = (k_1, \mathbf{k})\}$. By Lemma A1b $G|_{z=0}$ reduces each $l_2(\Lambda_{\mathbf{k}})$ so that $\Gamma|_{z=0}$ does also. Using the Leibniz formula

$$\frac{\partial^r}{\partial z^r}\Gamma_\Lambda = -\sum_{s=0}^{r-1}\binom{r}{s}\frac{\partial^r \Gamma_\Lambda}{\partial z^r}\frac{\partial^{r-s}G_\Lambda}{\partial z^{r-s}}\Gamma_\Lambda$$

and an induction argument the result follows.

Proof of Lemma 2.1a. By the polymer expansion of Ref. 3

$$\lim_{\Lambda\to Z^d} G_\Lambda(i,j,\{w_q = \beta\}, z = \beta) = G(i,j,B) = G(j - i = x, \beta)$$

uniformly for small $|\beta|$; β analyticity of G follows from analyticity of G_Λ given in Corollary A1 setting $\{w_q = \beta\}$, $z = \beta$. The bound on G follows from the same bound on G_Λ which in turns follows from Corollary A1 by a Cauchy estimate.

Proof of Lemma 1.2a. It can be shown that $\chi_\Lambda G_\Lambda \chi_\Lambda$ converges strongly to G in $l_2(Z^d)$ where χ_Λ is the operator of multiplication by the characteristic function of Λ. Using the Neumann series for Γ_Λ, $\chi_\Lambda\Gamma_\Lambda\chi_\Lambda$

converges strongly to Γ. The bound on Γ follows from the same bound on Γ_Λ which in turn follows from Corollary A3 by a Cauchy estimate. ■

The bound on $(\partial^2\Gamma/\partial\beta^2)(x=(\pm 1, \mathbf{0}), \beta)$ follows from the same bound on $(\partial^2\Gamma_\Lambda/\partial\beta^2)(i, j, \beta)$, $j = i + e_1$, $i_1 = p$. From Lemma A2a, b, suppressing the $i, j, \{w_q\}$ $q \neq p$, z arguments,

$$\Gamma_\Lambda(w_p) = \int_0^{w_p} \frac{\partial\Gamma_\Lambda}{\partial w_p'}(w_p')\,dw_p' = \int_0^{w_p}\left[1 + \int_0^{w_p'} \frac{\partial^2\Gamma_\Lambda}{\partial w_p''^2}(w_p'')\,dw_p''\right]dw_p'$$

setting all $\{w_q\}, z$ equal to β gives $\Gamma_\Lambda(\beta) = \beta + I(\beta)$ where

$$I(\beta) = \int_0^\beta dw_p' \int_0^{w_p'} \frac{\partial^2\Gamma_\Lambda}{\partial w_p''^2}(w_p'', \{w_p'' = \beta\}_{q\neq p}, z = \beta)\,dw_p''$$

$$= \int_0^\beta dw_p' \int_0^{w_p'} dw_p'' \int_0^{w_p''} dw_p''' \frac{\partial^3\Gamma_\Lambda}{\partial w_p'''^3}(w_p''', \{w_q = \beta\}_{q\neq p}, z = \beta)$$

where we have used Lemma A.2c. From this representation $I(\beta) = \beta^3 g(\beta)$, $g(\beta)$ analytic at $\beta = 0$, and since $\partial^2\Gamma_\Lambda/\partial\beta^2 = \partial^2 I/\partial\beta^2$ the result follows.

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